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Evaluation 3

Let $\mathcal{R}$ be the relation on $\mathbb{Z}$ defined by

$$x\mathcal{R}y \iff \exists k \in \mathbb{Z},\, x - y = 2k.$$

1. Prove that $\mathcal{R}$ is an equivalence relation. (3pt)

   - $\mathcal{R}$ is symmetric: If $x\mathcal{R}y$ then there exists $k \in \mathbb{Z}$ such that $x - y = 2k$. Therefore $y - x = 2 \times (-k)$. Since $k \in \mathbb{Z}$, we have $-k \in \mathbb{Z}$ therefore there exists a $k' = -k \in \mathbb{Z}$ such that $y - x = 2k'$. Hence $y\mathcal{R}x$.
   - $\mathcal{R}$ is reflexive: $\forall x \in \mathbb{Z}, x - x = 0 = 2 \times 0$. Since $0 \in \mathbb{Z}$, we have $x\mathcal{R}x$.
   - $\mathcal{R}$ is transitive: If $x\mathcal{R}y$ then $\exists k \in \mathbb{Z}$ such that $x - y = 2k$; if $y\mathcal{R}z$ then $\exists k' \in \mathbb{Z}$ such that $x - y = 2k'$. Hence $(x - y) + (y - z) = x - z = 2k + 2k' = 2(k + k')$. Since $k, k' \in \mathbb{Z}$, we have $\tilde{k} = k + k' \in \mathbb{Z}$. Therefore exists an integer $\tilde{k}$ such that $x - z = 2\tilde{k}$. Which implies that $x\mathcal{R}z$.

   Hence $\mathcal{R}$ is an equivalence relation.

2. What are the elements of $\mathbb{Z}/\mathcal{R}$? (3pt)

   - $\mathbb{Z}/\mathcal{R} = \{\mathcal{R}(a), a \in \mathbb{Z}\}$, where $\mathcal{R}(a) = \{b \in \mathbb{Z}, b\mathcal{R}a\}$.
   - $\mathcal{R}(0) = \{b \in \mathbb{Z}, b\mathcal{R}a\} = \{b \in \mathbb{Z}, \exists k \in \mathbb{Z}, b = 2k.\}$ which corresponds to the even numbers.
   - $\mathcal{R}(1) = \{b \in \mathbb{Z}, b\mathcal{R}a\} = \{b \in \mathbb{Z}, \exists k \in \mathbb{Z}, b = 2k + 1.\}$ which corresponds to the odd numbers.
   - Since the union of even numbers and odd numbers is equal to $\mathbb{Z}$ and since $\mathbb{Z}/\mathcal{R}$ is a partition of $\mathbb{Z}$, we deduce that $\mathbb{Z}/\mathcal{R} = \{\mathcal{R}(0), \mathcal{R}(1)\}$.

3. Prove that $\mathcal{R}(0) \cap \mathcal{R}(1) = \emptyset$? (2pt) The quotient set of a set A is a partition of A, then the intersection of two elements of the quotient set is always $\emptyset$.

4. What is $\inf\{\mathcal{R}(0), \mathcal{R}(1)\}$ and $\sup\{\mathcal{R}(0), \mathcal{R}(1)\}$ for the order relation $\subseteq$ on $2^{\mathbb{Z}}$? (3pt)

   - General statement: If E is a set and $A, B \in 2^E$, since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $A \cap B$ is a lower bound of $\{A, B\}$ for the order relation $\subseteq$. Let L be a lower bound of $\{A, B\}$ for the order relation $\subseteq$ then $L \subseteq A$ and $L \subseteq B$ which implies $L \subseteq A \cap B$. Hence $A \cap B$ is the greatest lower bound, that is $\inf(\{A, B\}) = A \cap B$. With this general statement we obtain $\inf\{\mathcal{R}(0), \mathcal{R}(1)\} = \mathcal{R}(0) \cap \mathcal{R}(1) = \emptyset$.
   - General statement: If E is a set and $A, B \in 2^E$, since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $A \cup B$ is an upper bound of $\{A, B\}$ for the order relation $\subseteq$. Let L be an upper bound of $\{A, B\}$ for the order relation $\subseteq$ then $A \subseteq L$ and $B \subseteq L$ which implies $A \cup B \subseteq L$. Hence $A \cup B$ is the least upper bound, that is $\sup(\{A, B\}) = A \cup B$. With this general statement we obtain $\sup\{\mathcal{R}(0), \mathcal{R}(1)\} = \mathcal{R}(0) \cup \mathcal{R}(1) = \mathbb{Z}$.

5. Let $f : \mathbb{Z} \to \mathbb{Z}/\mathcal{R}$.

   (a) Prove that f is never bijective. (2pt). Let $f : \mathbb{Z} \to \mathbb{Z}/\mathcal{R}$. Either $f(0) = \mathcal{R}(0)$ and since f is injective, $f(0) \neq f(1)$ implies $f(1) = \mathcal{R}(1)$ or $f(0) = \mathcal{R}(1)$ and since f is injective, $f(0) \neq f(1)$ implies $f(1) = \mathcal{R}(0)$. Since f is injective, $f(2) \neq f(0)$ and $f(2) \neq f(1)$ Which is not possible beacause $f(2) \in \{\mathcal{R}(0), \mathcal{R}(1)\} = \{f(0), f(1)\}$. Hence f cannot be injective.

   (b) Prove that if f is such that $\exists x, y \in \mathbb{Z}, f(x) \neq f(y)$ then f is surjective. (2pt) Either $f(x) = \mathcal{R}(0)$ and $f(x) \neq f(y)$ implies $f(y) = \mathcal{R}(1)$ or $f(x) = \mathcal{R}(1)$ and $f(x) \neq f(y)$ implies $f(y) = \mathcal{R}(0)$. In both case, $\mathbb{Z}/\mathcal{R} = f(\{x, y\}) \subseteq f(\mathbb{Z})$. Since we have always $f(\mathbb{Z}) \subseteq \mathbb{Z}/\mathcal{R}$ we deduce $f(\mathbb{Z}) = \mathbb{Z}/\mathcal{R}$ Which implies the surjectivity of f.

   (c) Find a function $f : \mathbb{Z} \to \mathbb{Z}/\mathcal{R}$ which is neither injective nor surjective. (2pt) Let $f : \mathbb{Z} \to \mathbb{Z}/\mathcal{R}$ defined by $\forall a \in \mathbb{Z}, f(a) = \mathcal{R}(1)$. $f(\mathbb{Z}) = \{\mathcal{R}(1)\} \neq \mathbb{Z}/\mathcal{R}$, then f is not surjective. By question 5.a, f cannot be injective. Hence f is neither injective nor surjective.